A MULTI-OBJECTIVE APPROACH TO APPROXIMATE THE STABILIZING REGION FOR LINEAR CONTROL SYSTEMS

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Abstract: Stability is a crucial issue to consider for control system design. In this paper a new multi-objective approach to approximate the stabilizing region for linear control systems is proposed. The design method comprises two stages. In the first stage, a bi-objective sub-problem is solved: the algorithm aims to calculate the vertices that maximize both the volume of the decision space and the percent of stable individuals generated within the decision space. In the second stage, the information gathered during the first stage is used to solve the actual multi-objective control design problem. To evaluate the proposed method a PID design problem is considered. Results show that in this case, our method is able to find better Pareto approximations than the classical approach.

1 INTRODUCTION

Multi-Objective Evolutionary Algorithms (MOEA) have been succesfully applied to control applications, provided the optimization problem can not be solved by classical methods (Fleming, 2001),(Tan et al., 2005). However, these algorithms often operate in a fixed search space predefined *a priori*, which hopefully contains the stabilizing region (Khor et al., 2002). Note that this requirement is hard to fulfil in practical control applications, since in general there is little information about the geometry of the stabilizing region. However, the stability of a control system is extremely important and is generally a safety issue.

Evolutionary computation techniques assume the existence of an efficient evaluation of the feasible individuals. However, there is no uniform methodology for evaluating unfeasible ones. In fact, several techniques have been proposed to handle constraints within the framework of MOEAs. The simplest approach is for example to reject unfeasible solutions (Michalewicz, 1995). In this work, we propose to determine an approximation of the stabilizing region trying to generate and to keep feasible individuals during the whole search process. Note that in practice the information about the stabilizing region is very important for the control engineer.

Quite surprisingly, there exist few references about MOEAs which consider a dynamic search space. As an example, a multi-population algorithm called Multi-Objective Robust Control Design (MRCD) was proposed in (Herreros, 2000). The search space is dynamic and a filtered elite is employed to preserve the best results for future generations.

In (Arakawa et al., 1998) an algorithm called Adaptive Range Genetic Algorithm (ARange GA) was described. The searching range is adapted in such a way that there is no need to define the initial range and nor the number of bits to determine the precision of results, if binary codification is adopted.

In (Khor et al., 2002) an inductive/deductive learning approach for single and multi-objective evolutionary optimization was proposed. The method is able of directing evolution towards more promising search regions even if these regions are outside the initial predefined space. For problems where the global optimum is included in the initial search space, it is able of shrinking the search space dynamically for better resolution in genetic representation.

Another algorithm called ARMOGA (Adaptive Range Multi-Objective Genetic Algorithm) was proposed in (Sasaki et al., 2002). Based on the statistics of designs computed so far, the new decision space is determined. The authors claim that their algorithm makes possible to reduce the number of evaluations to obtain Pareto solutions.

In other hand, randomized techniques for analysis of uncertain systems have attracted considerable interest in recent years, and a significant amount of results have appeared in the literature (Calafiore and Dabbene, 2006). The basic idea is to characterize uncertain parameters as random variables, and then to evaluate the performance in terms of probabilities. Analogously, probabilistic synthesis is aimed at determining the parameters so that certain desired levels of performance are attained with high probability.

In the following, a new randomized multi-objective approach to determine an approximation of the stabilizing region for linear controllers is presented. The design method comprises two stages. In the first stage a bi-objective sub-problem is solved: the vertices that maximize both the volume of the decision space and the ratio of feasible individuals are calculated. In the second stage, the information gathered during the first stage is used to solve the actual multi-objective control design problem.

To evaluate the proposed method a **PID** design problem is considered. Note that these controllers are still the preferred structure in most industries. And this is because of several reasons. From a practical view, they come pre-programmed in most commercial controllers and are well understood by both engineers and technicians. Thus, it is important to verify whether any design problem can be solved using a **PID** structure, before testing more complex solutions.

In the framework of this paper, we have another important motivation to consider **PID** controllers. Recently, Hohenbichler *et al* (2008) developed a **MATLAB®** toolbox, called **PIDROBUST**, which allows to calculate the stability region of control loops which contain an ideal **PID** controller. This is why we propose to evaluate the quality of the the boundaries which are going to be estimated in this paper by means of the new proposed randomized multi-objective approach.

The paper is organized as follows. In section 2, the design problem is formulated. In section 3 the proposed design method is described. In section 4, numerical results are given and finally in section 5 conclusions are given.

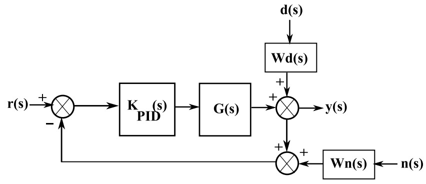

Figure 1: Control loop with a PID controller.

2 PROBLEM FORMULATION

Consider the control loop in figure 1. Therein, the ideal **PID** controller has the equation shown in (1). Blocks $W_n(s)$ and $W_d(s)$, $s \in \mathbb{C}$ represent arbitrary filters. The signals $r(s), d(s)$ and $n(s)$ represent the reference, disturbance and noise signals respectively. For regulation problems it is assumed $r(s) = 0$ and then equations (2), (3) and (4) hold.

$$K_{PID}(s) = \frac{K_D s^2 + K_P s + K_I}{s} \quad (1)$$

$$y(s) = -T(s)W_n(s)n(s) + S(s)W_d(s)d(s) \quad (2)$$

$$T(s) = \frac{K_{PID}(s)G(s)}{1 + K_{PID}(s)G(s)} \quad (3)$$

$$S(s) = \frac{1}{1 + K_{PID}(s)G(s)} \quad (4)$$

The transfer functions $S(s)$ and $T(s)$ are known as sensitivity function and complementary sensitivity function respectively. To attenuate disturbance and noise signals, both $S(s)$ and $T(s)$ must be made "small". However, note that from equations (3) and (4) it is impossible to decrease the norms of both $S(s)$ and $T(s)$ simultaneously: this is why we need to seek the best trade-off solutions. For simplicity, in this work only two $\mathcal{H}_2$ objectives are considered and the control problem can be stated as follows:

$$\min_{K_P, K_D, K_I \in \mathbb{R}} \begin{pmatrix} \|W_d S(K_P, K_D, K_I)\|_2 \\ \|W_n T(K_P, K_D, K_I)\|_2 \end{pmatrix} \quad (5)$$

subject to

$$K_{PID} = \frac{K_D s^2 + K_P s + K_I}{s} \in \mathcal{K}$$

where $\mathcal{K}$ is the set of stabilizing controllers.

3 DESIGN METHOD

To solve (5) a common approach is to define a penalty function to avoid unstable individuals. For example, a penalty function $p_i(K_{PID})$ $i \in \{1,2\}$ can be defined for each objective function as:

$$p_i(K_{PID}) = \begin{cases} f_i(K_{PID}) \text{ if } \max\{\mathrm{Re}[\lambda(K_{PID})]\} < -\mu \\ \eta * \max\{\mathrm{Re}[\lambda(K_{PID})]\} + B, \text{ otherwise} \end{cases} \quad (6)$$

where $\lambda(K_{PID})$ are the poles of $S(K_{PID})$, μ, η, B are fixed parameters and

$$f_1(K_{PID}) = \|W_d S(K_P, K_D, K_I)\|_2 \quad (7)$$

$$f_2(K_{PID}) = \|W_n T(K_P, K_D, K_I)\|_2 \quad (8)$$

Although simple, the penalty approach can be ineficient and lead to premature convergence. We propose instead to estimate an approximation of $\mathcal{K}$ (see the flowchart shown in figure 2). The proposed design method comprises two stages. During the first stage the goal is to calculate the vertices that maximize both the volume of the decision space and the percent of stable individuals. Note that these two objectives are in conflict.

Let n be the number of decision variables for problem (5) and $\overline{L}, \underline{L} \in \mathbb{R}^n$ be vectors defining the range for each variable. As an example, for the **PID** problem, we have $n = 3$ and

$$\begin{aligned} K_P &\in [\underline{L}_1, \overline{L}_1] \\ K_I &\in [\underline{L}_2, \overline{L}_2] \\ K_D &\in [\underline{L}_3, \overline{L}_3] \end{aligned} \quad (9)$$

The following sub-problem is considered during the first stage:

$$\min_{\overline{L}, \underline{L} \in \mathbb{R}^n} \begin{pmatrix} g_1(\overline{L}, \underline{L}) \\ g_2(\overline{L}, \underline{L}) \end{pmatrix} \quad (10)$$

where

$$g_1(\overline{L}, \underline{L}) = 1 - \frac{N_s(\overline{L}, \underline{L})}{N(\overline{L}, \underline{L})} \quad (11)$$

$$g_2(\overline{L}, \underline{L}) = \frac{V_{\max} - \prod_{i=1}^{n} (\overline{L}_i - \underline{L}_i)}{V_{\max}} \quad (12)$$

$$V_{\max} = \max_{\overline{L}, \underline{L} \in \mathbb{R}^n} \prod_{i=1}^{n} (\overline{L}_i - \underline{L}_i)$$

and

$$N_s(\overline{L}, \underline{L}) = \sum_{i=1}^{N(\overline{L}, \underline{L})} S_t(K_{PID_i}) \quad (13)$$

$$S_t(K_{PID}) = \begin{cases} 0 \text{ if } \max\{\mathrm{Re}[\lambda(K_{PID})]\} \geq -\mu \\ 1 \text{ otherwise} \end{cases} \quad (14)$$

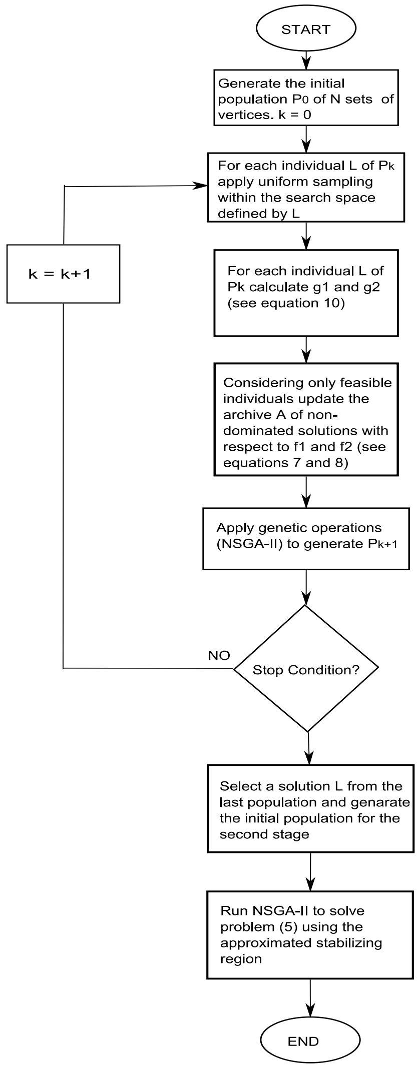

Figure 2: Flowchart of the proposed design method.

In (11) N_s is the number of stable individuals obtained via uniform sampling with boundaries $\overline{L}, \underline{L}$. The sample size $N(\overline{L}, \underline{L})$ may be calculated using the following Chernoff bound:

$$N(\overline{L}, \underline{L}) = \frac{1}{2\varepsilon^2} \ln \frac{2}{\delta} \quad (15)$$

It can be shown that if $N(\overline{L}, \underline{L})$ is taken according to (15) and the sampling is uniform the following equation holds:

$$\mathrm{Prob}\left\{\left| p - \frac{N_s}{N} \right| \leq \varepsilon\right\} \geq 1 - \delta \quad (16)$$

where

$$p = \mathrm{Prob}\{S_t(K_{PID}) = 1\} \quad (17)$$

We have fixed ε and δ to 0.1. Note that during the first stage an archive of non-dominated solutions can be updated, to be used later. In both stages any available MOEA can be used to solve the optimization problem. In this work we have chosen **NSGA-II** (Deb et al., 2000), which is a "state of the art" multi-objective genetic algorithm. **NSGA-II** is an improved version of the original algorithm proposed in (Srinivas and Deb, 1994). Note that this algorithm was previously used to design **PID** structures in (Lagunas, 2004). It has the following features:

- It uses an elitist principle.
- It uses and explicit diversity preserving mechanism.
- There is no sharing parameter to select.
- The sorting mechanism is faster than MOGA.

The offspring is created using the parent population and usual genetic operators. Thereafter, the two populations are combined together and a non-dominated sorting mechanism is used to classify the entire population.

Special mutation and cross-over operators were designed to generate individuals that remain within the limits of the (approximated) feasible region.

4 NUMERICAL RESULTS

Consider the open-loop model taken from (Hohenbichler and Abel, 2008):

$$G_0(s) = \frac{-0.5s^4 - 7s^3 - 2s + 1}{s^6 + 11s^5 + 46s^4 + 95s^3 + 109s^2 + 74s + 24} \quad (18)$$

and the filters W_n and W_d are defined:

$$W_n(s) = \frac{1}{s+1}, W_d(s) = 1 \quad (19)$$

For this plant, figure 3 shows an example of the convex stability polygons corresponding to $K_P = 1$. **PIDROBUST** allows also to plot the whole stabilizing region, as shown in figure 4.

The proposed design method was coded in **MATLAB®** 7.0. Figure 5 shows a typical Pareto approximation (Coello et al., 2007) obtained at the end of stage 1. Note that the Pareto front approximation shows that there exist no compromise solution between the two objectives. In this work the following solution have been arbitrary chosen:

$$\begin{aligned} [\underline{L}_1, \overline{L}_1] &= [-18.3, 6.1] \\ [\underline{L}_2, \overline{L}_2] &= [-4.3, 9.69] \\ [\underline{L}_3, \overline{L}_3] &= [-83.3, 14.3] \end{aligned} \quad (20)$$

which corresponds to $g_1 \approx 0.2$ and $g_2 \approx 0.99$.

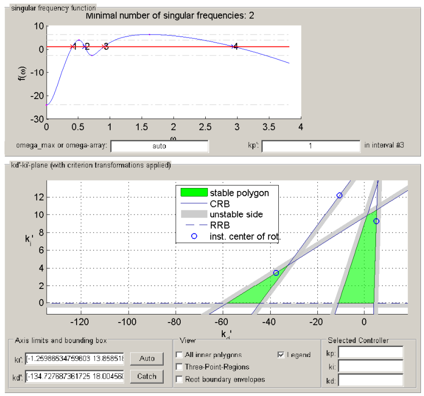

Figure 3: Example of the convex stability polygons corresponding to $K_P = 1$.

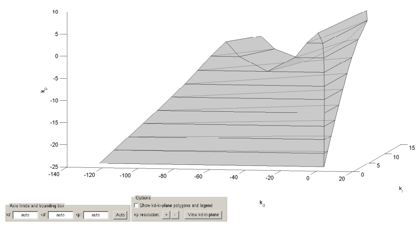

Figure 4: Stabilizing region.

To compare the results, the design problem was solved by means of the classical "penalty function based approach (in this case **NSGA-II** as a standalone algorithm), with $\mu = 10^{-4}, \eta = 10^3, B = 10^3$ and generating the individuals within the cube defined by:

$$K_P, K_D, K_I \in [-200, 200] \quad (21)$$

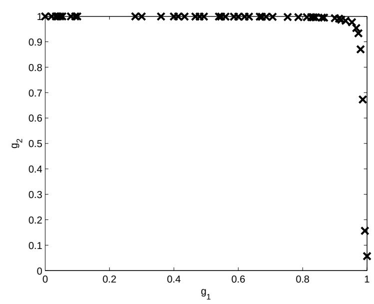

Figure 5: Pareto approximation obtained at the end of the first stage.

The comparison is fair because this region envelopes the true stabilizing region, although this information is not available *a priori*. We also include the results corresponding to the true stabilizing region obtained by **PIDROBUST**, with boundaries:

$$\begin{aligned} [\underline{L}_1,\overline{L}_1] &= [-22.5,4.9] \\ [\underline{L}_2,\overline{L}_2] &= [0,12.59] \\ [\underline{L}_3,\overline{L}_3] &= [-118,5.24] \end{aligned} \quad (22)$$

Thus, three algorithms are compared:

- **A1** : NSGA-II with $K_P,K_D,K_I \in [-200,200]$.
- **A2** : NSGA-II: Stage 1 and Stage 2 (proposed new method) with $\varepsilon = 0.1$ and $\delta = 0.1$
- **A3** : NSGA-II + PIDROBUST (using the true stabilizing region).

Figures 6, 7 and 8 show typical approximations obtained via **A1**, **A2** and **A3** respectively. In the multi-objective optimization framework, there are in general three goals (i) the distance of the resulting nondominated set to the Pareto-optimal front should be minimized, (ii) a uniform distribution of the solutions found is desirable, and (iii) the extent of the obtained nondominated front should be maximized. We have chosen four indicators to evaluate each Pareto approximation:

- Set Coverage (C). Given two Pareto approximations $\mathcal{PF}_1$ and $\mathcal{PF}_2$, the Set Coverage is defined as follows:

$$C(\mathcal{PF}_1,\mathcal{PF}_2) = \frac{|\{b \in \mathcal{PF}_2, \exists a \in \mathcal{PF}_1 : a \preceq b\}|}{|\mathcal{PF}_2|} \quad (23)$$

$C(\mathcal{PF}_1,\mathcal{PF}_2)$ measures the proportion of elements in $\mathcal{PF}_2$ that are dominated by at least one element in $\mathcal{PF}_1$.

- Efficient Set Space (ESS). Defined as follows:

$$ESS = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(d_i - \overline{d})^2} \quad (24)$$

Hereby, d_i denotes the minimal Euclidean distance from the image $\mathcal{PF}_1(x_i)$ of a solution $x_i, i = 1,\ldots,N$, to the true Pareto front, and

$$d_i := \min_{\substack{j=1,\ldots,N \\ i\neq j}} d_{ij} \quad (25)$$

$$\overline{d} := \frac{1}{N}\sum_{i=1}^{N} d_i, \quad (26)$$

where d_{ij} is the Euclidean distance between $\mathcal{PF}_1(x_i)$ and $\mathcal{PF}_1(x_j)$

- Hypervolume (HV). This indicator measures the volume covered by the approximation with respect to a reference point.
- Max. Distance (MD). Defined as:

$$MD = \max_{\substack{i,j=1,\ldots,N \\ i\neq j}} d_{ij} \quad (27)$$

The setting used for each algorithm is shown in table 1. Tables 2 and 3 show the indicators values for 30 executions. The mean value and the standard deviation (between brackets) is presented for each indicator. Table 4 shows the results of Mann-Whitney-Wilcoxon tests with a significance level $\alpha = 0.05$. Therein, the symbols $=,\Uparrow,\Downarrow$ mean that the algorithm in the corresponding row is statistically equal, better or worse than the algorithm in the column, with respect to the four indicators C, ESS, HV and MD.

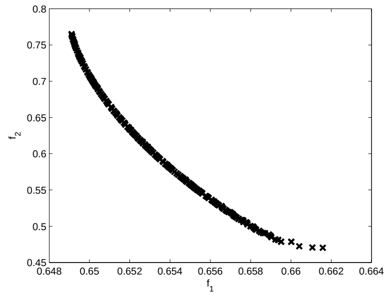

Figure 6: Example of Pareto approximation obtained with algorithm A1.

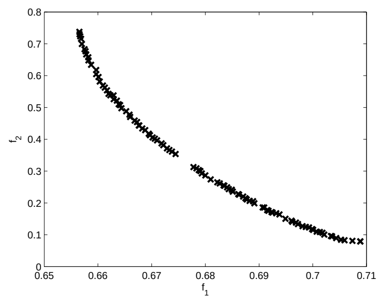

Figure 7: Example of Pareto approximation obtained with algorithm A2.

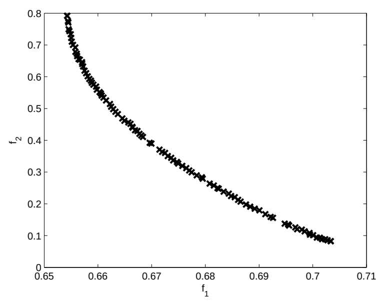

Figure 8: Example of Pareto approximation obtained with algorithm A3.

Table 1: Parameter Settings NSGA-II.

Representation	Real numbers
Cross-Over Operator	Uniform
Cross-Over Probability	0.9
Mutation	Gaussian Perturbation
Mutation Probability	0.1
Stop Condition A1	300 generations
Selection Scheme A1	(200+200)
Stop Condition A2,A3	50 generations
Selection Scheme A2,A3	(100+100)

Table 2: Results for the coverage indicator.

$C(A_i,A_j)$	**A1**	**A2**	**A3**
A1	–	0.26(0.31)	0.23(0.33)
A2	0.26(0.43)	–	0.20(0.33)
A3	0.23(0.39)	0.56(0.35)	–

Table 3: Results for indicators ESS, HV and DMAX.

	ESS	*HV*	*MD*
A1	0.0452(0.05)	0.03(0.03)	0.03(0.03)
A2	0.14(0.08)	0.13(0.00)	0.15(0.09)
A3	0.11(0.07)	0.10(0.02)	0.11(0.07)

Table 4: Results for Mann-Whitney-Wilcoxon tests with a significance level $\alpha = 0.05$.

	A1	**A2**	**A3**
A1	–	$=,\Uparrow,\Downarrow,\Downarrow$	$=,\Uparrow,\Downarrow,\Downarrow$
A2	$=,\Downarrow,\Uparrow,\Uparrow$	–	$\Downarrow,\Downarrow,\Uparrow,\Downarrow$
A3	$=,\Downarrow,\Uparrow,\Uparrow$	$\Uparrow,\Uparrow,\Downarrow,\Uparrow$	–

5 CONCLUSIONS

In this paper a new design method for solving multi-objective control problems was described. Results show that, for the PID control problem, the proposed method **(A2)** is able to find better approximations than the conventional method (**A1**) with respect to the *HV* and *MD* indicators, given the same number of functions evaluations. Moreover, when compared to **A3** (the algorithm using the true stabilizing region), results show that **A2** is indeed able to find a good approximation of the feasible region.

In the future the authors will try to extend this work, in order to consider more complex control problems, with more design variables and objectives. For that goal, a more sophisticated representation of the search space is needed (stage 1), in order to make the sampling process more efficient.

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