Geometric gravitational origin of neutrino oscillations and mass-energy

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Abstract

A mass-energy scale for neutrinos was calculated from the null cone curvature using geometric concepts. The scale is variable depending on the gravitational potential and the trajectory inclination $i$ with respect to the field. The proposed covariant neutrino equation provides the adequate curvature scalar. The curvature mass-energy at the Earth surface varies from a horizontal value $E_0^h = 0.402$ eV to a vertical value $E_0^v = 0.569$ eV.

Earth spinor waves with winding numbers $n$ show differences $\Delta E_{nm}^2$ within ranges $2.05 \times 10^{-3}$ eV$^2 \leq E_{10}^2 (i) \leq 4.10 \times 10^{-3}$ eV$^2$ and $3.89 \times 10^{-5}$ eV$^2 \leq \Delta E_{21}^2 (i) \leq 7.79 \times 10^{-5}$ eV$^2$. These waves interfere and the different phase velocities produce neutrino-like oscillations.

The experimental results for atmospheric and nuclear neutrino oscillation mass parameters respectively fall within the $E_{10}$ and $E_{21}$ theoretical ranges. Neutrinos in outer space, where interactions may be neglected, appear as particles travelling with zero mass-energy on null geodesics. These gravitational curvature energies are consistent with neutrino oscillations, zero neutrino rest masses and Einstein’s General Relativity and energy mass equivalence principle. When analyzing or averaging experimental neutrino mass-energy results of different experiments on the Earth it is of interest to consider the possible influence of the trajectory inclination angle.

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1- Introduction

Some geometric ideas about space-time may shed light on one important present-day problem in particle physics: the neutrino mass. It has been known for a long time that light deflection experiments confirm that the null cone acquires curvature under gravitation as predicted by Einstein’s General Relativity. The curvature of geodesics affects particle trajectories as mass does. It may be interpreted using the energy-mass equivalence principle that particles acquire a curvature mass-energy determined by the square root of the gravitational potential absolute value. This means that weak gravitational effects should not be neglected “a priori” when studying very fast particles moving on trajectories on the curved null cone or nearby mass hyperboloids. In this article we consider the motion of a neutrino wave by discussing its differences with a light wave instead of simply assuming geodesic motion on a flat null cone. Geometric concepts and calculations are presented in appendices.

Rather than only rely on the metric structure of space-time, Einstein and Schrödinger maintained that it is necessary to start from fundamental affine connections in order to describe the interactions experienced by matter as it follows geometric equations of motion. The connection transformations form a group with subgroups and should unify different interactions. A modern formulation of these ideas is by a local physical action of a generalized connection or potential $A$ as indicated in appendix A. The connection determines the straightest line between two points. If this line is also the shortest distance between the points the connection is called Riemannian. The group of the connection may be determined by the space-time Clifford algebra which acts over a generalized spin space. The connection determines a general curvature which is not necessarily Riemannian. When the interaction reduces to a gravitational field the geometric group reduces to SL(2,C), designated as L, which is homomorphic to SO(3,1) and the connection is pseudo-Riemannian.

In general the equation of motion of matter on space-time is a generalized covariant Dirac equation. Under reduction to the SL(2,C) group in vacuum this equation reduces to a covariant Weyl equation for a zero bare-mass left-handed fundamental spinor. The spinor evolution or motion may still be described as motion under a connection as E. Cartan envisioned. This may be accomplished by a spinor connection valued on the sl(2,C) subalgebra and its corresponding curvature.

In his book E. Cartan proves a theorem on the impossibility of maintaining the geometric interpretation of spinors when using the original relativistic arbitrary system of curved coordinates. We should use the Cartan orthonormal moving frame formulation to display the fundamental importance of spinor geometry on relativity.

2- The effects of Curvature

We shall use a simple example to illustrate the influence of curvature on physical motion. Consider the motion of light along a null geodesic on the Earth surface. For the weak Schwarzschild gravitational field we may neglect the $\varphi^2$ terms. The metric interval for a radial null path may be approximately written in terms of a dimensionless Lorentzian-section curvature $K$, as discussed in appendix D,

$$d\tau^2 = (1 + 2\varphi)dt^2 - \frac{dr^2}{(1 + 2\varphi)} - r^2\left(d\theta^2 + d\phi^2 \sin^2 \theta\right) = 0$$

$$dt^2 - dr^2 \approx -2\varphi\left(dt^2 + dr^2\right) = K\left(dr^+\right)^2$$
where $r^+ = \int d^2 (t+r)$ is the affine parameter along the null-geodesic. This equation (2) may also be written in a form similar to the energy-momentum relation for a particle with a mass proportional to the line curvature $\kappa$,
\[
\left( \frac{dt}{dr^+} \right)^2 - \left( \frac{d\rho}{d\rho^+} \right)^2 = \left( \sqrt{-2\varphi} \right)^2 = K \equiv \kappa^2.
\]

The gravitational field determines the Gaussian curvature $K$ of a null Lorentzian 2-surface in a pseudo-Riemannian space as indicated in appendix D. In similarity with curves on spheres the trajectories on a curved 2-surface in the curved null cone are characterized by their line curvatures $\kappa$ which are determined by $K$ and may be related to mass-energy. The line curvatures $\kappa$ of the two principal orthogonal geodesic directions in a 2-surface obey
\[
K(p) = \kappa_1 \cdot \kappa_2.
\]

Consider the spacelike wave vector $k$ of a zero-mass particle (see fig. 1). There is a 2-dimensional null flat space spanned by the timelike $t$ and the spacelike $s$ unit vectors. This plane is the $t \sim s$ boost space, which contains the trajectory four-velocity $u$ and is tangent to a Lorentzian null surface. There is a well defined curvature tensor on this surface. The Gaussian curvature may be generalized on manifolds provided with moving frames and special affine transformation groups on surfaces. Physically a Lorentzian curvature form represents the inertial action density of space-time geometric transformations. We let the physical problem determine the significant curvature involved.

On a Lorentzian surface the scalar product is not geometrically significant because it is zero for any pair of null tangent vectors. Instead we should use the geometrically significant fact that Lorentzian surfaces are characterized by pairs of null directions parametrized by the time $t$ or a space coordinate $s$ along the trajectory. We relate the curvature form to the null directions which are determined by $t$ equal $s$. A null direction is characterized by a point on the unit sphere $S^2$. The unit sphere $S^2$ is a cross section of the null cone at $t = -t$ and corresponds to the celestial sphere. Unit vectors and directions are characterized by points on the unit sphere $S^2$. Points on a null cone are characterized by points on spheres of radius $r = t$ related by Lorentz dilation transformations. The observation of wave propagation is the projection of this structure on the geometry of the observer. This projection should provide a significant null-section curvature scalar $\tilde{K}$.

The dimensionless null-section curvature scalar $\tilde{K}$ for a null 4-vector (spin 1) on an Earth trajectory in space-time determined by this projection is calculated in appendix D. The curvature scalar depends on the dimensionless weak gravitational potential ratio $\varphi$ and increases with the trajectory inclination angle $\iota$ relative to the $s = h$ horizontal plane orthogonal to the $s = r$ radial (vertical) direction,
\[
\tilde{K}_s \leq \varphi (1 + \sin^2 \iota) \leq \tilde{K}_r.
\]

3- An even sl(4, $\mathbb{R}$) Spinorial Neutrino Equation.

There is a fundamental relation of 2-spinors with the space-time formulation of relativity, which is a remarkable geometric correspondence showing the fundamental spinor structure of light propagation. The light null-cone vectors may be considered as the square of spinors. In fact any single 2-spinor $\zeta$ defines a real null light-like vector $\lambda$ by the quadratic relation
\[
\lambda^\mu = \zeta X^A \sigma_\mu \zeta^A.
\]
Consider now the equation of motion for a spinor. The spinor motion may be described as a particular case of the covariant Dirac equation of motion. This is accomplished by a connection valued on the sl(4,R) algebra and its corresponding curvature. An even sl(4,R) generator determines a complex structure on space-time as indicated in appendix C. We should distinguish the geometric spinor structure from its position complex coordinates on a surface in space-time. We designate the complex coordinate adapted to the spinor motion by \( w \). The direction of motion, a null 4-velocity vector \( u \), should be given in relation to the orthonormal set \( \kappa^\mu \) of the Clifford algebra in order to define the null Lorentzian subspace implicit in the spinor motion. The derivatives with respect to the two complex coordinates \( w \) and \( z \) should also be related to the orthonormal set of the Clifford algebra. To preserve both the spinor and complex structures we should project the complex coordinate derivatives onto an orthonormal set adapted to the direction. In order to do this, we should define a set in the \( \mathbb{R}_{\gamma I} \) Clifford algebra adapted to null vectors and complex coordinates in terms of the orthonormal set,

\[
\begin{align*}
\kappa^z &= \frac{1}{\sqrt{2}} \left( \kappa^z + i \kappa^y \right) \\
\kappa^- &= \frac{1}{\sqrt{2}} \left( \kappa^z - i \kappa^y \right) \\
\kappa^w &= \frac{1}{\sqrt{2}} \left( \kappa^w + i \kappa^l \right) \sim \frac{1}{\sqrt{2}} \left( \kappa^w + \kappa^l \right) \\
\kappa^\pi &= \frac{1}{\sqrt{2}} \left( \kappa^w - i \kappa^l \right) \sim \frac{1}{\sqrt{2}} \left( \kappa^w - \kappa^l \right).
\end{align*}
\]

The projection of the neutrino derivative on the neutrino wave vector or equivalently on the tangent plane represented by \( \kappa^\mu \) determines a coordinate invariant scalar operator. Using \( \hbar = 1 \) physical units the neutrino equation of motion is

\[
\kappa^A \nabla_A \nu = 0.
\]

Since there is no motion along \( z \) the equation of motion may be simply written

\[
\kappa^A \nabla_A \nu = \kappa^w \nabla_w \nu + \kappa^\pi \nabla_\pi \nu = 0.
\]

If we take the second derivative there appear quadratic operators of the form.

\[
\kappa^A \kappa^B \nabla_A \nabla_B \nu = 0.
\]

The \( \kappa^i \) anticommuting properties and the curvature symmetries determine that there are no contributions from mixed components. These terms introduce the even 4 by 4 sl(4,R) Pauli matrix representations. Decomposing the second derivative in its symmetric and antisymmetric parts we obtain using the commutation relations

\[
\frac{1}{2} \left\{ \kappa^A, \kappa^B \right\} \nabla_A \nabla_B \nu + \frac{1}{2} \left[ \kappa^A, \kappa^B \right] \nabla_A \nabla_B \nu = \partial_w \partial_\pi \nu + \kappa^0 \kappa^8 \tilde{\Omega}_{w\pi} \nu = 0.
\]

In the Schwarzschild geometry the sl(2,C) complex curvature form \( \Omega \) is defined by the well known Schwarzschild curvature tensor or form \( \Omega \). The form components are equal, with the vector so(3,1) generator and the real volume element replaced by the spinor sl(2,C) generator and the complex volume element, as shown on appendix C according to their 1/2 homomorphic structures. Equation (3) determines a single null-section curvature scalar component by contraction of the curvature tensor with the physical components of the null vector. The single curvature scalar corresponding to a spinor trajectory on a null section along \( s \) is, in terms of the Newtonian potential \( \varphi \),

\[
\frac{1}{2} \kappa = \tilde{K} \equiv \kappa^0 \kappa^8 \Omega dw \wedge d\bar{w} \left( t \frac{\partial}{\partial w}, \left| t \frac{\partial}{\partial \bar{w}} \right| \right).
\]
On the Earth surface the wave vector \( k \) defines a polar angle \( \theta \) with respect to the gravitational field direction, which determines the Lorentzian section of interest in the null-cone subspace. We obtain

\[
\mathcal{K} = (\mathcal{W} \cos^2 \theta + \mathcal{W} \sin^2 \theta) \, dt \wedge d\bar{w} \left\{ \frac{t}{\partial w}, \left( \frac{t}{\partial \bar{w}} \right) \right\}.
\]

Because of the relation of time with the radius of the dilating wave spheres this gives, in terms of the inclination angle \( \iota \) relative to a horizontal plane,

\[
\mathcal{K} = \left( \frac{\varphi}{2r^2} \sin^2 \iota + \frac{\varphi}{4r^2} \cos^2 \iota \right) \, dt^2 = \frac{\varphi}{2} \sin^2 \iota + \frac{\varphi}{4} \cos^2 \iota
\]

\[
\mathcal{K} = \frac{\varphi}{2} = 2 \mathcal{K}_f
\]

These dimensionless curvatures are relative Schwarzschild curvatures\(^{12}\) or ratios with respect to the curvature of the neutrino wave dilating spheres \( s \Omega \) on the null cone. The neutrino curvature component \( \varphi \Omega \) in eq. (3) has dimensions \( L^{-2} \) and is proportional to the relative \( K \). The factor of proportionality should be the fundamental curvature \( \varphi \Omega \) or squared geometric mass \( m^2 \), indicated in appendices A and F

\[
\frac{\Omega}{s \Omega} = K = \frac{\mathcal{K}}{\mathcal{K}_f} = \frac{m^2}{2m^2} = \frac{\mathcal{K}_f}{\mathcal{K}} = \frac{\mathcal{K}_f}{2} = \frac{\mathcal{K}_f}{2}
\]

\[
-\Delta \nu + \mathcal{W} (\partial_w, \partial_{\bar{w}}) \nu = -\tilde{\Delta} \nu + 2m^2 \tilde{K} \nu = 0.
\]

This result may also obtained using the matrices of the \( R_{1,3} \) Clifford Algebra in eq. (2). The Laplace-Beltrami operator on the \( w \bar{w} \) complex surface in the complex \( TM \) bundle is proportional to the neutrino physical 4-momentum of dimensions \( L^{-2} \),

\[
\tilde{\Delta} \nu = -\left( \frac{\partial}{\partial w} \frac{\partial}{\partial \bar{w}} \right) \nu = 2m^2 \tilde{K} \nu = \nu P^2 \nu,
\]

consistent with a Laplacian on a null surface in the \( TM \) bundle,

\[
\Box \nu = \frac{1}{4} \left( \left( \frac{\partial}{\partial t} \right)^2 - \left( \frac{\partial}{\partial s} \right)^2 \right) \nu = 2m^2 \frac{1}{2} \nu = \nu P^2 \nu.
\]

The Laplace operator is mapped to a neutrino Klein-Gordon energy-momentum operator \( \nu P^2 \). If we restore the Planck constant the neutrino energy operator is \( \frac{P}{m} \partial_t \). Planck’s energy relation \( E = h \nu \) is satisfied for the natural spinor frequency which is half the vector frequency because the spinor linear group \( SU(2) \) volume is twice the volume of the orthogonal group \( SO(3) \).

We discuss further the differences in curvature because their role in determining mass-energy. The tangent space-time \( TM \), with its complex structure is a curved space (it is really a curved fiber bundle). The curvature operators are base-space 2-forms valued on their respective group Lie algebras as indicated in appendix A,

\[
\Omega = E_j \mathcal{W}_j.
\]
Since both elements, $E^j_i$ and $\Omega^j_i$, in the curvature correspond to bivectors related to rotations or 2-forms, the symmetry of the resultant curvature tensor components (between the first and last index pairs) corresponds to an interchange of the rotational roles played by $E^j_i$ and $\Omega^j_i$. This means that both elements, $E^j_i$ and $\Omega^j_i$, are linear combinations of bivectors related to infinitesimal rotations or generators in their respective sl(2,C) or so(3,1) representations.

The infinitesimal group transformation $\sigma_a$ on a spin-$\frac{1}{2}$ (spinor) representation has a relative $\frac{1}{2}$ factor with respect to the usual spin-1 (vector) representation transformation as indicated in appendix B. Similarly the complex 2-form $dz \wedge d\bar{z}$ which determines the volume (area) element on a 2-dimensional complex subspace also introduces a relative $\frac{1}{2}$ factor with respect to the usual volume element on a 2-dimensional vector subspace as indicated in appendix C. Due to the quadratic dependence of the curvature on these two bivectors there are two $\frac{1}{2}$ factors among the respective curvatures. A dimensionless Lorentzian-section curvature $K$ given on $TM$ determines a spinor-section curvature $\bar{K}$,

$$\frac{\frac{1}{2}K}{\bar{K}} = \left(\frac{1}{2}\right)^2 = \left(\frac{\kappa}{\kappa}\right)^2.$$

There is a relative $\frac{1}{2}$ factor between the principal line curvatures $\kappa$ in these spaces. The line curvatures of the principal geodesic directions affect particle trajectories as mass. These simple relations are in accordance of the more detailed geometric calculations.

### 4- Neutrino Mass-energy.

In a previous work discussed in appendix F it was proposed that the leptons correspond to topological excitation waves characterized by a winding (or rather wrapping) topological number $n = 0, 1, 2$ which may determine leptonic flavor levels. We calculated the lepton bare mass ratios in terms of algebraic volume ratios of the group symmetric subspaces. Similarly we may define characteristic potential energy scale factors for neutrino trajectory curvature ratios. An energy factor also determines the line curvature $\kappa$ corresponding to the trajectory of the spin-1 vector representation or photon on the null-space. The energy factor may be calculated from the dimensionless curvature scalars.

The different relative potential energy factors for each representation trajectory determine proportional line curvatures $\kappa$. For the spin representation which is given by the sl(2,C) connection instead of the homomorphic so(3,1) connection. The dimensionless line curvatures which determine spinor wave parameters are given by eq. (4),

$$\kappa = \sqrt{\frac{-\varphi}{2}},$$

$$\kappa = \frac{-\varphi}{2} \sqrt{\sqrt{\frac{gR}{\sqrt{2}}}} = \frac{\sqrt{9.8066 m/s^2 \times 6.3781 \times 10^8 m}}{\sqrt{2 \times 2.9979 \times 10^8 m/s}} = 1.319 \times 10^{-5}.$$
We obtain the energy factors from the Gaussian curvature ratios using the volumes of the same previously used symmetric spaces related to the $S(M)$ spinor bundle. In particular the geometric factor $V(C_R)$, indicated in appendix F, is the factor of proportionality of the electron bare rest mass with respect to $\varpi$. This factor gives the physical dimension of $\varpi$ in terms of the electron mass,

$$\frac{m_\nu}{m_e} \approx \frac{\kappa}{C_R}.$$ 

On the Earth surface this effect is higher than expected because it depends on the square root of the dimensionless gravitational potential ratio $\varphi$. The numerical value on the Earth surface may be taken as a constant of the order of the expected electron neutrino mass,

$$\frac{\kappa}{2M} = \frac{m_e\sqrt{-\varphi}}{\sqrt{2V(C_R)}} = \frac{3m_e\sqrt{-\varphi}}{16\pi \sqrt{2}}.$$ 

The different horizontal curvature scalar is a half of the radial curvature scalar because the Schwarzschild metric curvature scalar is not isotropic (see appendix D). It gives the minimum value of mass-energy for a horizontal neutrino trajectory on the Earth surface,

$$rE_0 = \frac{m_e\sqrt{-\varphi}}{\sqrt{2V(C_R)}} = \frac{5.11 \times 10^3 \times 2.638 \times 10^{-5}}{16\sqrt{2\pi}3} = 0.569 \text{ eV} \sim m_\nu.$$ 

The action of the geometric potential $A$ (sl(2,C) connection) on the covariant Weyl wave equation is similar to the action of rotation generators on gravitational precession experiments. The SL(2,C) group acts on its spin representations. In particular we are considering representations induced by its SU(2) subgroup, designated as H, which are spinor functions on the SL(2,C)/SU(2) coset. This 3-dimensional coset space is related to the null cone. The SL(2,C) neutrino representations have zero rest mass and are geometrically related to the massive leptons through the $\pi_j$ homotopy groups of SL(2,C), Sp(4,R) and SU(2) which are isomorphic to the integers $\mathbb{Z}$ as indicated in appendix F. Some of these theoretical curvature effects were first reported within the context of neutrino velocity determinations.

### 5- Squared Energy Effects due to Winding Numbers.

The characteristic mass-energy scale $E_0$ equal to this minimum value. Mass-energy increases with the trajectory inclination angle $\iota$ relative to a horizontal plane.

If gravitation is neglected there may be significant errors in the calculation of small rest masses at very-high-energy processes. The errors would be of the order of the energy $E_0$ related to the linear curvature $\kappa$ as indicated in appendix E. Some of these theoretical curvature effects were first reported within the context of neutrino velocity determinations.
equations on the images which we call channels \( n = 0, 1, 2 \). There are varied curvature scalars \( K_n \) which correspond to the neutrino channels.

Since all states are equivalent under the SU(2) group they equally share the curvature energy at any given point and direction on the static space-time. The number of energy-degenerate group states in channel space \( n = 0 \) is the group volume \( V(H) = V_0 \). The \( n \)-wound neutrino wave functions correspond to an increased volume \( V(n) \) and number of states because of the additional channels available. The squared mass-energy density per state, which is proportional to the curvature \( K \), should be a characteristic parameter of any particular neutrino channel wave.

The variation of the volume, which is the only variable, together with the presence of non-zero curvature may produce a variation of the neutrino energy per state associated to the equation source term. There are varied energy values \( E_n \) which correspond to the neutrino channel source terms. As a model we may consider that there are neutrino energy currents along channels or parallel paths in space-time. Each parallel path corresponds to a neutrino of different type. The connection action on the channels \( \nu_n \) simultaneously determines the energy values and trajectories. We may express the energy when \( n \) varies from zero as a function

\[
E^2(n) = \frac{Km^2}{(V(C_R))^2} \left( \frac{V_0}{V(n)} \right)^2 = \frac{E^2_0V^2_0}{V^2(n)}.
\]

The volume of the spin subgroup \( V(H) \) represents a geometric inertial opposition to the energy flow. This is due to the H group action, directly on the L/H coset, which corresponds to an inverse action of the group H on itself which passes to the divisor \( H^{-1} \). These actions, the trajectories and curvature scalars for \( n = 1, 2 \) waves vary discretely from their \( n = 0 \) values. The variation when other channels are present must be due to the varied number of states under the action of the \( Z \) homotopy group and may be of order \( V^{-1} \) between levels.

The SU(2) group \( H \) acts through the connection as a derivative. We may assume that the action of the homotopy group \( Z \) produces a variation of the curvature from its \( n = 0 \) value by a variation with respect to the number of states or geometric group volume \( V \). We define a map from the homotopy group \( Z \) to a discrete subset of difference operators \( \Delta \) in the set of variation operators \( \delta \) with a derivative as operation,

\[
Z = (0, 1, 2, 3 \cdots n) \rightarrow \left\{ \Delta^0_v \equiv \frac{\delta^n \left( I/V^2 \right)}{\delta V^n} \right\},
\]

\[
\left( \Delta^0_v, \Delta^1_v, \Delta^2_v \cdots \Delta^n_v \right) = \left( \frac{I}{V^2}, -(I, -2V^{-1}, (-I)^2 6V^{-2}, \cdots (-I)^n (n + 1)!V^{-n} \right).
\]

The resultant difference operators \( \Delta^i_v \) determine discrete \( \Delta K_n \) curvature variations and directly squared energy differences \( \Delta E^2 \) in eq. (2) for waves \( \nu_v \) in each channel \( n = 1, 2 \),

\[
\Delta^0_v(K) = K_0 = E^2_0,
\]

\[
\Delta^1_v(K) = \frac{-2K_0}{V} \Rightarrow \Delta K_{i0} = \frac{-2K_0}{V} = \frac{-2E^2_0}{V} = \Delta E^2_{i0},
\]

\[
\Delta^2_v(K) = \frac{6K_0}{V^2} \Rightarrow \Delta K_{ij} = \frac{6K_0}{V^2} = \frac{6E^2_0}{V^2} = \Delta E^2_{ij}.
\]
The 3 neutrino $n = 0, 1, 2$ waves interfere due to their small mass-energy differences $\Delta E_{mn}$ producing neutrino oscillations. Observed very small experimental neutrino masses and neutrino oscillations may be caused by this gravitational effect. Instead of constant neutrino masses we really would have variable neutrino curvatures and energies determined by the null cone geometry produced by gravitation.

A wave current in level $n$ should imply currents in all lower levels $l \leq n$. Due to the proportionality of the equations, the energy in the channels up to winding number $n = 2$ and their differences satisfy the relations

$$E_i^2 = E_0^2 + E_{2i}^2 = E_0^2 \left(1 - \frac{2}{V(SU(2))}\right),$$

$$E_2^2 = E_i^2 + E_{2i}^2 = E_0^2 \left(1 - \frac{2}{V(SU(2))} + \frac{6}{V^2(SU(2))}\right) \approx E_0^2 \left(1 - \frac{2}{V(SU(2))}\right) \approx E_i^2$$

$$\Delta E_{0i}^2 = E_0^2 - E_i^2 = \frac{2E_0^2}{16\pi^2} \approx \Delta E_{02}^2,$$

$$\Delta E_{2i}^2 = E_2^2 - E_i^2 = \frac{6E_0^2}{(16\pi^2)^2}.$$

The potential neutrino energies $E_n(\nu)$ may be found from eqs. (6, 9, 10). In particular, the neutrino “dressed” mass in experiments on the Earth surface would depend on its spacelike direction because the Schwarzschild metric null-section curvature scalar is not isotropic. We express the numerical energy results in terms of the trajectory inclination angle $i$ (see fig. 1) within their horizontal minimum and vertical maximum, using the standard Earth gravitational values which may be taken as constants. We have, respectively, for Earth neutrinos associated with the short-range energy $E_{0i}^2 \approx E_{02}^2$ or the long-range energy $E_{2i}^2$,

$$2.05 \times 10^{-3} \text{ eV}^2 \leq \Delta E_{0i}^2(i) = \frac{2E_0^2(1 + \sin^2 i)}{16\pi^2} \leq 4.10 \times 10^{-3} \text{ eV}^2,$$

$$3.89 \times 10^{-5} \text{ eV}^2 \leq \Delta E_{2i}^2(i) = \frac{6E_0^2(1 + \sin^2 i)}{(16\pi^2)^2} \leq 7.79 \times 10^{-5} \text{ eV}^2.$$

It may be considered that these neutrino states with gravitational energy determined by eqs. (11, 12) are energy eigenstates with wavefunctions $\nu_n$. They are characterized by the winding numbers and may be taken as the neutrino mass states of the standard theory of neutrino oscillations.

In general we can say that the neutrino mass-energy relations would be variable, depending on the gravitational potential in each experiment. A neutrino is created by a nuclear reaction in a definite state. From this initial state neutrino waves are driven and completely determined by the null cone curvature scalar lens effect until they are destroyed by a final nuclear collision. If the curvature scalar $K$ is zero, neutrino waves in all channels move equally free on a flat null cone. Neutrinos in outer space, where interactions may be neglected, would have zero line curvatures and would appear as massless particles travelling as photons on flat null geodesics.
Atmospheric neutrino experiments\textsuperscript{20} indicate a $2.4 \times 10^{-3}$ eV\textsuperscript{2} result, 17% higher than the $2.05 \times 10^{-3}$ eV\textsuperscript{2} minimum energy value of the short-range-oscillation. An average trajectory inclination $i = 24^\circ$ may produce a variation of the curvature scalar $K$ and $E_i^o(t)$ sufficient to account for the difference.

The reactor and solar neutrino experiments indicate a $7.6 \times 10^{-5}$ eV\textsuperscript{2} combined result\textsuperscript{21}, 2% lower than the $7.79 \times 10^{-5}$ eV\textsuperscript{2} maximum value of the long-range-oscillation. The solar neutrino wave mass-energy $E_i^o(t)$ has a diurnal anisotropy due to the Earth rotation. There is a variation of the curvature scalar $K$ which may sufficiently increase the observed $\Delta E_i^{(2)}(t)$ beyond the $5.84 \times 10^{-4}$ eV\textsuperscript{2} average value.

6- Conclusions.

The covariant equation (2) determines the neutrino wave function from the null-section curvature scalar $K$. This curvature scalar is determined by equation (4) from the gravitational potential. These dimensionless factors are related to energy-momentum by a geometric mass.

We calculated the neutrino mass-energy from the line curvature $\kappa$ using dimensionless ratios of volumes of symmetric SL(4,R) coset spaces, previously used to calculate particle bare-rest-mass ratios. The compact subgroup SU(2) of the gravitational SL(2,C) group determines variations of the curvature scalar $K$ among the topological neutrino states which define fundamental squared energy differences. These squared energy differences determine oscillations among the neutrino states.

The experimental results of neutrino mass-energy measurements and oscillations on trajectories appear to be consistent with these theoretical results and their physical interpretation without the need to assign bare rest masses to these particles.

In other words, gravitational curvature lens effects are consistent with null-cone neutrino oscillations. There is no clear experimental distinction between mass-energy in the neutrino motion on a curved null cone and mass in its motion on a nearby hyperboloid in flat space-time.

The neutrino curvatures and energies determined by the geometry produced by gravitation are really variables. These results are due to the presence of the potential term $2\varphi(r)$ in the space-time Schwarzschild metric. At the Earth surface the dimensionless radius of curvature of null cone trajectories is of order $10^7$. When averaging experimental mass-energy values it is necessary to consider the possibility of dependence with respect to trajectory inclination. If there are additional weak fields on space-time there may be a higher effective potential $\varphi$ and the numerical results may increase. In general for stronger fields a unified geometry would produce stronger particle effects.

Bohr might have been correct when he presented the idea that neutrinos may be related to gravitation, as indicated in appendix E. A neutrino is created by a nuclear reaction in a definite state. From this initial state neutrino waves are driven and consistently determined by the null cone curvature scalar until they are destroyed by a final nuclear collision.

We may have looked for the graviton in the wrong place, associated to the metric. The metric may be given by a vector frame determined by squared spinors as indicated in section 3. If we look at the space where the gravitational curvature and potential act we find the spin-$\frac{1}{2}$ neutrino frame field $\nu$. Instead of a metric wave we have a neutrino wave with energy proportional to the frequency, as shown by equation (5), which may be considered the real graviton. This is consistent with the fact that gravitational and neutrino fields are the only ones not blocked by the Earth crust.
This work is in line with previous theoretical work. Our method is not based directly on the metric properties of space-time but rather on the affine properties of matter transformations as its excitations evolve in space-time. The generalized theory, presented over a period of years, is based on geometric ideas introduced by Einstein, Cartan and Schrödinger. Further theoretical analysis and experiments may be required to fully understand the physical effects of the spinor geometry associated to particles.

Appendix A

Traditionally General Relativity has been expressed using differential geometry. General coordinates are conveniently used to find solutions, specially for static and stationary gravitational fields. On the other hand, mathematicians moved away from general coordinates looking for a more geometrical approach to a modern differential geometry. In particular, Elie Cartan elaborated the method of the “repere mobile” or moving frames. A frame is an ordered basis for a vector space. A moving frame is a function whose values are frames in the various tangent spaces \( M_p \) at points \( p \) in a manifold \( M \). A moving frame is not necessarily related to a proper coordinate system since the frame vectors need not commute. Later Koszul introduced a (Koszul) connection which is a function \( D \) associated to vector fields which satisfies certain axioms. This is a generalization for the classical connection in a coordinate system. Finally Ehresmann completed the modern concept of connections using the action of a Lie group on the geometric elements. The use of gauge fields in Physics is another example of the applications of these geometric ideas to generally non Riemannian spaces.

In parallel to this development, Cartan also introduced the notion of spinors. This concept related to Clifford algebras was applied to quantum physics by the introduction of the Pauli and Dirac matrices and later to General Relativity and Gravitation by Penrose.

A unified relativistic theory of gravitation, electromagnetism and other interactions was developed along these ideas. The geometric structure group \( SL(4,R) \) of the connection is determined by the space-time Clifford algebra which acts over a generalized spin space. The field equation is a generalized Maxwell equation. When the interaction reduces to only a gravitational field, the geometric group \( SL(4,R) \) reduces in a limit to \( SL(2,C) \) which is homomorphic to \( SO(3,1) \), the connection is pseudo-Riemannian and the tensor calculus reduces to the standard Ricci calculus. The geometric and algebraic modernization was a long process and it would be inappropriate to present a treatment in this article. We should limit ourselves to present the fundamental references.

The connection \( \omega = A \) is a 1-form valued in the Lie algebra. The curvature \( \Omega \) is a 2-form valued in the Lie algebra which may be expressed as

\[
\Omega = E^i_i \Omega^j_j
\]

in terms of a standard basis \( E \) in the Lie algebra and standard 2-forms \( \Omega \). There is a fundamental geometric inverse length or mass-energy \( \mathcal{M} \) which may be defined in terms of the connection,

\[
\mathcal{M} = \frac{1}{2} \text{tr} \left( J^i J^j A_i A_j \right) \equiv J \star A.
\]

Mass may be defined in an invariant manner in terms of this mass-energy, depending on the connection and the matter-spinor moving frames. There is a geometric background solution where \( \mathcal{M} \) is a constant. This constant is also related to the generalized curvature \( \Omega \) and the Newtonian gravitational constant by the equation

\[
\partial_i \partial^i \phi = \frac{4 \pi}{\mathcal{M}} \lim_{\varepsilon \to 0} \rho(\varepsilon) \equiv 4 \pi G \rho.
\]
Appendix B.

We present here an explicit display of the rotational transformation differences between vectors and spinors using, for convenience, a language close to relativistic physics rather than Cartan’s geometric language.

The Lorentz SL(2,C) connection and covariant derivative in spinor space define the SO(3,1) connection and covariant derivative in curved space-time. The morphism between the connections is determined by the homomorphism between the Lorentz SO(3,1) and SL(2,C) groups. In general, the six connection 1-forms shown in appendix D correspond to the generators which form a basis in the Lie algebra of both groups. In particular, we now display this relation using the connecting Pauli matrices $\sigma$, 

$$l_\mu = \text{tr} \left( \sigma^\mu g^\dagger \sigma_i g \right) \quad l_\mu \in \text{SO}(3,1); g \in \text{SL}(2,C).$$

The one dimensional spinor subgroup generated by $\sigma_3$ is

$$\Lambda = \exp(\beta \sigma_3) = I \cosh \beta + \sigma_3 \sinh \beta = I + \beta \sigma_3 + O(\beta^2) + \cdots$$

and the corresponding Lorentz one-dimensional subgroup is

$$L^\mu = \text{tr} \left( \sigma^\mu \exp(\beta \sigma_3) \sigma_i \exp(\beta \sigma_3) \right)$$

which may be written as a $t-z$ subspace transformation

$$L^\mu_{\nu} = \begin{pmatrix} \cosh 2\beta & 0 & 0 & \sinh 2\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh 2\beta & 0 & 0 & \cosh 2\beta \end{pmatrix} = L^\mu_{\nu} + 2\beta \cosh \beta + O(\beta^2) + \cdots \approx L^\mu_{\nu} + 2\beta \sigma_3^\nu.$$

It is clear that this spinor generator $\sigma_3$ corresponds to the Lorentz boost generator whose action on the asymptotic light-cone zone dilates the time and distance coordinates on the cone. The amount of deformation produced by the corresponding SO(3,1) connection form $\omega_\beta^\mu$ along the null geodesic is twice the amount of the deformation produced by the SL(2,C) connection form $\sigma_3$ on the original spinor $z$, because it also includes the deformation of the conjugate spinor. Equal expressions hold for the three spinor generators $\sigma_i$ in the non-compact boost sector of the algebra. Similar but antisymmetric expressions, with the hyperbolic functions replaced by the circular functions, hold for the three spinor generators $i\sigma_i$ in the compact rotation sector. These relations are not due to a pure coincidence, they are determined by the homomorphism of the related groups associated to the gravitational connection. In general the group transformations generated by the connections are related by the 2 to 1 relationship between the two indicated connection group transformations.

The SL(2,C) and SO(3,1) groups are 2 to 1 homomorphic. This means that the parameter space for SL(2,C) is twice the size of the parameter space for SO(3,1). A single transformation in SO(3,1) is equivalent to a pair of transformations in SL(2,C) and actually splits in two transformations: a spinor transformation and its similar conjugate transformation. The corresponding transformation produced by the Cartan-Ehresmann connection on a single spin-1/2 representation has a relative 1/2 factor with respect to the usual spin-1 vector representation transformation. The same relation respectively applies to their compact subgroups U(1) and SO(2). A factor of 2, well known for vector and spinor rotations, is introduced by the $4\pi$ volume of U(1) which is twice the volume of SO(2) and
affects the definition of spinor frequency \( f \). It also is a common factor for the non-compact Lorentz group generators, in particular for those which generate the time dilations on the light cone and the boosts inside the cone.

**Appendix C**

The even \( \text{sl}(4,\mathbb{R}) \) connection generators on a manifold \( M \) include antisymmetric tensors \( J \) of type \((1, 1)\) which obey \( J^\dagger = -I \) determining an almost complex structure on a real 4-vector space\(^{11} \). The corresponding complex vector space has 2 complex dimensions. This structure \( J \) determines an integral almost complex structure on \( TM \) and therefore a unique complex structure on the manifold. We indicate that the manifold with the complex structure \( M \) and the original \( M \) are related manifold representations of space-time with points locally identified by the physical space-time events. The curved complex manifold is a valid representation of space-time capable of representing a physical relativistic local expression in terms of time and space variables related to complex 2-spinors.

There should be a local bundle map \( j \), which allows a physical interpretation on \( M \) and the definition of complex coordinates associated to the coordinates on \( M \). The 2 to 1 mapping from 2-spinors to 4-vectors determines a local relation between the Lie algebras of the linear transformations on \( TM \) and \( TM \). Covariant equations under an \( \text{SL}(2,\mathbb{C}) \) connection on the curved spinor complex bundle provide an expression for the spinor motion.

In order to work with any spacelike direction \( s \) on space-time \( M \) we use Minkowskian \( c=1 \) coordinates \( t = i t \), \( x^i = s \), \( x^i = x \), \( x^i = y \) with \( s \) along the neutrino motion. The complex differential form basis and its dual complex vector basis may be obtained by pulling back these coordinates. Thus we may define complex coordinates \( z^i \) on the 2-dimensional curved manifold with index summation also including conjugate indices.

If we have a spinor frame field we have a section \( \eta \) in the spinor bundle \( TM \) and a map \( j \) given by the neutrino current

\[
j : M \rightarrow M
\]

\[
z' = \text{Re} z' + \text{Im} z' \equiv j^s s + i (j^l l) \equiv w
\]

\[
z^2 = \text{Re} z^2 + \text{Im} z^2 \equiv j^x x + ij^y y \equiv z
\]

The \( dw \wedge d\bar{w} \) subspace is a tangent Lorentzian surface adapted to the neutrino motion. The volume element on the complex\(^{11} \) spinor subspace spanned by \( w \) which corresponds to the \( t \sim s \) boost space is, omitting the complex coordinate indices,

\[
\sqrt{\Sigma} = dw \wedge d\bar{w} = (j^s ds + ij^l dl) \wedge (ds - ij^l dl) = -2i(j^s ds \wedge j^l dl) = 2(ds \wedge dt) = 2 \sqrt{\Sigma}.
\]

This means that the volume (area) element on this spinor complex tangent subspace determines twice the volume (area) element on the corresponding 2-dimensional vector tangent space. The dual complex vector base\(^{11,28} \) on this complex null subspace is

\[
\frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial (j^s)} i \frac{\partial}{\partial (j^l)} \right)
\]

\[
\frac{\partial}{\partial \bar{w}} = \frac{1}{2} \left( \frac{\partial}{\partial (j^s)} + i \frac{\partial}{\partial (j^l)} \right).
\]

The desired \( \text{sl}(2,\mathbb{C}) \) valued spinor curvature \( \hat{\nabla} \) and connection forms on the \( dw \wedge d\bar{w} \) spinor subspace of \( TM \) corresponding to any \( dx^a \wedge dt \) space-time subspace may be obtained
from the vector curvature and connection forms given in appendix D by replacing the vector boost generators and forms with the corresponding spinor generators and forms. For the radial and horizontal directions we obtain

\[ \Omega^1 = \frac{2\varphi}{r^3} E_i^0 j^* dr \wedge j^* dt = \frac{2\varphi}{r^3} E_i^0 j^* dl \wedge j^* dr \rightarrow \bar{\Omega} \sin^{-1} \theta = \frac{2\varphi}{r^3} \left( \frac{\kappa^\rho \kappa^s}{2} \right) \left( \frac{dw \wedge d\overline{w}}{2} \right) \]

\[ \Omega^2 = \frac{\varphi}{r^2} E_i^0 j^* ds \wedge j^* dt = \frac{\varphi}{r^2} E_i^0 j^* dl \wedge j^* ds \rightarrow \bar{\Omega}_2 = \frac{\varphi}{r^2} \left( \frac{\kappa^\rho \kappa^s}{2} \right) \left( \frac{dw \wedge d\overline{w}}{2} \right) \]

\[ \Omega^3 = \frac{\varphi}{r^2} E_i^0 j^* ds \wedge j^* dt = \frac{\varphi}{r^2} E_i^0 j^* dl \wedge j^* ds \rightarrow \bar{\Omega}_3 = \frac{\varphi}{r^2} \left( \frac{\kappa^\rho \kappa^s}{2} \right) \left( \frac{dw \wedge d\overline{w}}{2} \right) \].

These 2-forms give the dimensionless Gaussian curvature scalar associated to the radial and the two horizontal sl(2,C) spinor generators on a 2-dimensional \( dw \wedge d\overline{w} \) subspace,

\[ \mathcal{K} = \frac{K}{4}. \]

The Gaussian curvature may be generalized on manifolds provided with moving frames and special affine transformation groups on surfaces\(^6\). Physically a Lorentzian curvature form represents the inertial action density of space-time geometric transformations. We let the physical problem determine the significant curvature.

**Appendix D**

We obtain the curvature for a weak Schwarzschild space-time in terms of the Newtonian gravitational potential \( \varphi \) using Minkowskian coordinates with the fourth coordinate \( l = it, \ c = I \)

\[ dr^2 = -ds^2 = -g_{\mu\nu} dx^\mu dx^\nu = (1 + 2\varphi) dl^2 + \frac{dr^2}{(1 + 2\varphi)} + (dx^r)^2 + (dx^t)^2. \]

The relevant non-zero classical connection coefficients\(^3\) \( \Gamma^\delta_{\beta\gamma} \) (Christoffel symbols) may be calculated from the metric. The Cartan connection 1-forms \( \omega \), which correspond one-to-one to the Lorentz SO(3,1) generators are obtained from the orthonormal tetrad,

\[ d\Theta^\delta + \omega^\delta_{\beta} \wedge \Theta^\beta = 0 \]

\[ \theta^0 = \sqrt{1 + 2\varphi} dt = \sqrt{\frac{2GM}{c^2 R}} dt, \quad \theta^i = \frac{dx^i}{\sqrt{1 + 2\varphi}} \approx \sqrt{\frac{2GM}{c^2 R}} dr, \]

\[ \theta^2 = dx^2, \quad \theta^3 = dx^3. \]

The well known six Riemann curvature 2-forms are\(^4\),

\[ \Omega = D\omega = d\omega + \omega \wedge \omega \]

\[ \Omega^0_i = \Omega^0_j \approx \varphi'' dr \wedge dt + O(\varepsilon) \approx -\frac{2GM}{c^2 R^3} dr \wedge dt = 2\varphi \frac{dr \wedge dt}{R^2} \]

\[ \Omega^2_i \approx \varphi'' \frac{dx^2}{2} \wedge dt + O(\varepsilon) \approx -\frac{GM}{c^2 R^3} dx^2 \wedge dt = \varphi \frac{dx^2 \wedge dt}{R^3} \]

\[ \Omega^3_i \approx \varphi'' \frac{dx^3}{2} \wedge dt + O(\varepsilon) \approx -\frac{GM}{c^2 R^3} dx^3 \wedge dt = \varphi \frac{dx^3 \wedge dt}{R^3} \]
\[ \Omega_2^\prime = \Omega_2^\prime \approx -\varphi'' dx^2 \wedge dr + O(\varepsilon) \approx \frac{GM}{c^2 R^2} dx^2 \wedge dr = -\varphi \frac{dx^2 \wedge dr}{R^2} \]

\[ \Omega_3^\prime \approx -\varphi'' dx^3 \wedge dx^2 + O(\varepsilon) \approx \frac{2GM}{c^2 R^3} dx^3 \wedge dx^2 = -2\varphi \frac{dx^3 \wedge dx^2}{R^2} \]

\[ \Omega_4^\prime = \Omega_4^\prime \approx -\varphi'' dr \wedge dx^3 + O(\varepsilon) \approx \frac{GM}{c^2 R^2} dr \wedge dx^3 = -\varphi \frac{dr \wedge dx^3}{R^2} \]

These six forms correspond to the six SO(3,1) generators which may be denoted by the following matrices:

\[
E_0^\prime = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_0^\prime = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_0^\prime = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\]

\[
E_1^\prime = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_1^\prime = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_1^\prime = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

The Schwarzschild metric curvature is not isotropic. The curvature boost components along a radial direction or its orthogonal (horizontal) directions \( x^a \) may be written in dimensionless coordinates respectively defined by

\[ \Omega_\theta^\prime = \varphi E_\theta^\prime \frac{dr \wedge dt}{R^2} \]

\[ \Omega_\phi^\prime = \varphi E_\phi^\prime \frac{d\phi \wedge dt}{R^2} \]

\[ \Omega_\psi^\prime = \varphi E_\psi^\prime \frac{d\psi \wedge dt}{R^2} \]

The null geodesic section of interest is spanned by the wave vector \( k^a \) and the time direction \( t \) (see fig.1). Vectors in the directions of the wave vector and time define the null-section curvature scalar. The curvature form should be projected and evaluated along these vectors as indicated in section 2,

\[ \Omega_s^\prime = \sin t \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \cos t \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

The dimensionless curvature scalar is expressed as

\[ t^2 \left( -\varphi dr \left( \sin t \frac{\partial}{\partial r} + \cos t \frac{\partial}{\partial h} \right) \right) \left( \sin t \frac{\partial}{\partial t} + \cos t \frac{\partial}{\partial h} \right) \]

\[ t^2 \left( \varphi dr \left( \sin t \frac{\partial}{\partial t} + \cos t \frac{\partial}{\partial h} \right) \right) \left( \sin t \frac{\partial}{\partial t} + \cos t \frac{\partial}{\partial h} \right) \]

Because of the relation of time with the radius of the dilating wave spheres this reduces to
\[ K' = 2 \varphi \sin^2 \iota + \varphi \cos^2 \iota = \varphi (1 + \sin^2 \iota) . \]
The curvature scalar increases with the trajectory inclination angle \( \iota \) relative to a horizontal direction \( h \),

\[ K_h' \leq \varphi (1 + \sin^2 \iota) \leq K_r' . \]

**Appendix E**

In order to discuss the motion of a neutrino in the proper physical context we should first make some theoretical-historical considerations. Cartan introduced geometric spinors\(^5,29\), in 1913. Weyl\(^34\) soon developed a physical theory for a 2-component (Weyl) spinor. Pauli used the spin matrices to formulate the theory of non relativistic spin\(^35\) in 1927. Dirac developed the standard 4-component relativistic spinor theory for massive fermions in 1928.

It is now well known that if the particle has zero mass the Dirac equations decouple into 2-component Weyl zero mass \( \eta \) spinor equations\(^36\) which may be written in terms of the Pauli matrices \( \sigma \) and the Planck constant \( \hbar \) as

\[ i \hbar \overleftrightarrow{\sigma} \partial_{\mu} \eta = 0 . \]

Since the introduction of a (neutrino) particle\(^37\) in beta decay by Pauli in 1930 the main question debated was whether its mass was zero or very small. Traditionally the Weyl equation was used to describe the neutrino. Today the bound for the electron neutrino mass is estimated to be of the order of a few eV and questions arise about the adequate equation to use. It is also well known that the experimental mass of particles is generally explained by field energy corrections added to an assumed finite “bare” particle mass to obtain a theoretical “dressed” mass in the process of mass renormalization\(^38,39\) for Dirac’s equation. If the curvature effects of the gravitational potential term were equivalent to the effects of a small finite dressed mass-energy we similarly may explain the small neutrino mass starting from a zero “bare” neutrino mass. Or equivalently, the neutrino oscillations\(^40,41\) could also be explained by the dressed mass-energy contribution to a zero bare mass neutrino. Both points of view may be consistent with observed neutrino oscillations. Certainly these questions are still open for further experiments and theoretical ideas. From a theoretical point of view it appears that the Lee and Yang two-component theory of the neutrino\(^42\) using the zero mass Weyl equation, together with the Einstein’s energy-mass equivalence should be the fundamental ideas. When there is no gravitational field the photon curve becomes a flat space-time null geodesic. Under these conditions the neutrino, as determined by the standard Weyl equation, should also follow the same path of the photon. Nevertheless when there is a gravitational field and space is curved we expect\(^43\) the neutrino spinor field \( \nu \) to obey a geometric covariant equation determined from the integrability conditions of the full \( sl(4,R) \) field equation restricted to the gravitational spinor \( sl(2,C) \) connection 1-form \( \omega \). This would be in accordance with the old Bohr idea\(^44,45\) that the neutrino should be related to gravitation.

The general equation\(^46\) of motion reduces to a generalized covariant Weyl equation written in terms of the orthonormal basis \( \kappa^\nu \) of the \( R_{1,1} \supset sl(4,R) \) Clifford algebra,

\[ \kappa^\nu \nabla_{\mu} \nu = 0 . \]

This equation is a space-time image of a spinor covariant transplantation equation and determines a covariant Klein-Gordon 4-momentum equation on the spinor bundle \( S(M) \) over space-time.

There is a well established classical spinor analysis on curved spinor spaces and a curved space-time.\(^47,48,49,50\) Using the Cartan analysis the spinor connection may be expressed as \( sl(2,C) \) Lie algebra valued 1-forms. It is known that the three Pauli matrices over the
complex numbers may be considered as the six generators of the SL(2,C) group of transformations which is homomorphic to the SO(3,1) Lorentz group of transformations.

Appendix F

Mass may be defined in an invariant manner in terms of energy, depending on the connection and the moving matter frames (“repere mobile”) in a geometric theory. All bare rest masses depend on a geometric mass which may be normalized to the proton mass. In particular the proton to electron bare rest mass ratio is \( 6\pi^4 \). It has been shown that the quotient of the bare rest masses of all known leptons are essentially determined, up to correction terms of the order of the \( \alpha \) constant, to the quotient of geometric masses corresponding to subgroup excitations including algebraic and topologic effects. In general, the geometry determines the geometric excitation mass spectrum, which for low masses, essentially agrees with the physical particle mass spectrum. The masses of the leptons increase under the action of a strong connection (relativity of energy) and are related to meson and quark excitation masses. There are massive connection excitations whose masses correspond to the weak boson masses and allow a geometric interpretation of Weinberg’s angle. The necessary first order corrections, due to the interaction of the excitations, are of order of the \( \alpha \) constant, equal to the order of the discrepancies.

In particular it was also proposed that massive leptons are representations of the Sp(4,R) group by functions characterized by rest mass \( m \) and spin over the mass shell hyperboloid subspace while neutrinos are representations of the SL(2,C) group as functions characterized by helicity over the null hypercone subspace. The masses of leptons correspond to topological excitation waves characterized by a winding (or rather wrapping) topological number \( n = 0, 1, 2 \) which may determine leptonic flavor levels. The \( n \)-homotopy group of a group is defined as the classes of mappings from the \( n \)-spheres to the group space with the operation of juxtaposition of the sphere images. The third homotopy group \( \pi_3 \) of Sp(4,R), isomorphic to \( Z \), defines the winding or wrapping number \( n \).

The mass of leptons is proportional to the number of states (different momentum values) which is proportional to the volume of a total geometric \( C^T \) space related to the bundle \( S(M) \), which depends on the wrapping number \( n \) and may be calculated,

\[
V(C^T) = V(U(I)) \times V(C_R)^{n+1} \quad n \neq 0, \\
V(C_R) = \frac{16\pi^3}{3},
\]

where \( C_R \) is the space of relativistic inequivalent points of the coset \( \text{Sp}(4,R)/\text{SL}(2,C) \). The winding or wrapping number \( n \) determines the number of \( C_R \) images. The bare masses corresponding to the topological \( n \)-excitations are proportional to the volumes

\[
V(C^T) = 4\pi \left( \frac{16\pi^3}{3} \right)^{n+1} \quad 0 < n \leq 2,
\]

which may be expressed in terms of the electron bare mass \( m_e \).

For excitations with wrappings 1 and 2 we have the corresponding bare mass ratios of the \( \mu \) and \( \tau \) leptons. The lepton mass is determined by \( n \) on the mass shell hyperboloid and therefore \( n \) also characterizes lepton massive representations.
The neutrino and electron geometric mass-energy under a gravitational field are expected to have a small correction proportional to the field line curvature $\kappa$. The total mass-energy ratio of these two particles under gravitation would be

$$\frac{m_\nu}{m_e} = \left( \left( \frac{\text{SL}(2,C)}{\text{SL}(2,C)} \right)_R^V + \kappa \right) \mathcal{M} \approx \left( \frac{\text{SL}(2,C)}{\text{SL}(2,C)} \right)_R^V \left( \left( \frac{\text{SL}(4,R)}{\text{SL}(2,C)} \right)_R^V + 0 \right) \frac{\mathcal{M}}{C_R} = \frac{\kappa}{C_R}.$$
38 J. Schwinger, Editor, Selected Papers on Quantum Electrodynamics (Dover, N. Y.) (1958).
45 D. I. Blokhintsev and F. M. Galperin, Gipoteza neutrino i zakon sokhraneniya energii, Pod znamenem marxisma, # 6, p. 147.
Wave vector $k$
Wave frequency $\omega$
Inclination angle $\iota$
Radial direction $r$
Horizontal direction $h$
Time direction $t$
Velocity 4-vector $u$

$$u = \omega t + k_s \hat{s} = \omega t + k_r \hat{r} + k_h \hat{h}$$

Figure 1